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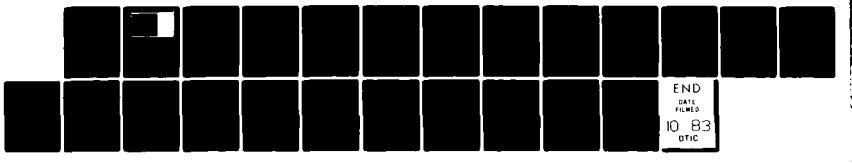

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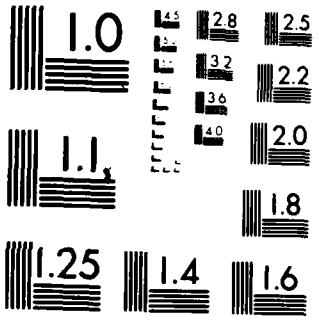

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A COLLOCATION METHOD FOR COMPUTING EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACIAN

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ABSTRACT

A boundary collocation method is presented for the computation of eigenvalues and eigenfunctions of the Laplacian on bounded simply connected plane regions with smooth boundary. Particular emphasis is placed on the selection of an appropriate function space and its basis for the approximation of the eigenfunctions. Numerical examples are presented.

AMS (MOS) Subject Classifications: 65N25, 65N35

Key Words: Eigenvalue, eigenfunction, Laplacian, boundary collocation

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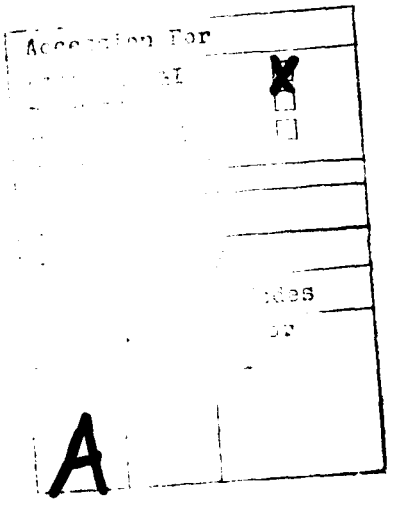
SIGNIFICANCE AND EXPLANATION

Several successful applications of the boundary collocation method for computing eigenvalues and eigenfunctions of the Laplacian on certain regions exist in the literature (see e.g. Fox, Henrici and Moler [5], and Moler [6]). However, Bates and Ng [1] reported that the method did not converge for some regions. We suggest a strategy for selecting collocation points, function space for the approximation of eigenfunctions, and basis of this function space. Our results apply to regions for which there are convergence difficulties with the methods referred to above.



A COLLOCATION METHOD FOR COMPUTING EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACIAN

Lothar Reichel

1. INTRODUCTION

Let Ω be a bounded simply connected plane open region with smooth boundary $\partial\Omega$, and let $\bar{\Omega} := \Omega \cup \partial\Omega$. We consider the computation of eigenvalues λ and eigenfunctions u of

$$(1.1) \qquad \begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Several successful applications of the boundary collocation method to solve this problem on certain regions exist in the literature, see e.g. Fox, Henrici and Moler [5], and Moler [6]. The purpose of this paper is to describe a strategy for choosing collocation points and function space for the approximation of eigenfunctions. We also discuss the selection of basis of the function space. The need for such a strategy is illustrated by the computational results of Bates and Ng [1]. Choosing function space and collocation points quite arbitrarily, they not seldom experience convergence difficulties and conclude [1], p. 153, 'The point matching or collocation method is an attractive economique technique from the point of view of programming and computer time [....]. Unfortunately the method is not always valid'. Nonconvex regions appear to be especially difficult.

We will represent eigenfunctions using an integral operator introduced by Vekua [9]. This operator maps functions analytic in Ω onto solutions of (1.1). The connection between eigenfunctions and analytic functions, to be described in detail in Section 2, implies that the approximation of an

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eigenfunction is equivalent to the approximation of a certain analytic function on Ω. The selection of function space for the approximation of eigenfunctions can be studied by considering the approximation of analytic functions in Ω. Approximation of analytic functions in Ω by rational functions with fixed poles is discussed in Section 3. When approximating analytic functions by rational functions with fixed poles using interpolation, the poles and interpolations points (= collocation points) should not be allocated independently. The connection between analytic functions and solutions of (1.1), as well as the reported difficulties in Bates and Ng [1], suggest that the choice of function space for the approximation of eigenfunctions and the selection of collocation points should not be done independently either. We begin by allocating collocation points on $\partial\Omega$, and then define a function space which depends on the distribution of collocation points. Our collocations scheme is presented in Section 4, which also contains a brief discussion on the computation of error bounds derived by Moler and Payne [7]. Numerical examples are found in Section 5.

2. AN INTEGRAL REPRESENTATION

This section summarizes some relevant results on the integral operator of Vekua, and on the smoothness of the eigenfunctions. Let Ω be as in Section 1, and let $C^{p,\gamma}(\bar{\Omega})$ denote the set of functions whose pth derivative is Hölder continuous on $\bar{\Omega}$ with Hölder constant γ. Let λ denote a real nonnegative constant. The following lemma is a special case of Theorem 4.2 of Eisenstat [3].

Lemma 2.1

Let $u \in C^2(\Omega) \cap C^{p,\gamma}(\bar{\Omega})$, $p + \gamma > 0$, satisfy

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega . \tag{2.1}$$

Then u has the representation

$$u(z) = \mathrm{Re}\Big(\phi(z) - \frac{\lambda}{2}(\bar{z}-\bar{z}_1)\int_{z_1}^{z} \phi(\zeta) J_1\big(\sqrt{\lambda(z-\zeta)(\bar{z}-\bar{z}_1)}\big)d\zeta\Big), \quad z \in \bar{\Omega} , \tag{2.2}$$

where $\phi(z)$ is a function analytic in Ω and $\phi \in C^{p,\gamma}(\bar{\Omega})$. $J_1(r)$ denotes the 1st order Bessel function of the 1st kind. ϕ is uniquely determined by the requirement $\mathrm{Im}\phi(z_1) = 0$. z_1 is an arbitrary but fixed point in $\bar{\Omega}$. $\bar{z}$ denotes the complex conjugate of z, and similarly for $\bar{z}_1$.

Conversely, if ϕ is analytic in Ω and $\phi \in C^{p,\gamma}(\bar{\Omega})$, then u defined by (2.2) satisfies (2.1) and $u \in C^2(\Omega) \cap C^{p,\gamma}(\bar{\Omega})$. ∎

The lemma shows that the approximation of an eigenfunction of (1.1) is equivalent to the approximation of an analytic function ϕ. If $\partial\Omega$ is analytic, then u solving (1.1) can be continued analytically across $\partial\Omega$, hence ϕ, defined by lemma 2.1, can be continued analytically across $\partial\Omega$. For nonanalytic boundary curves the following lemma gives the smoothness of u on $\partial\Omega$. The lemma is a special case of Theorem 7.1 of Eisenstat [3].

Lemma 2.2

Let $\partial\Omega \in C^{\max(p,1),\gamma}$ and let u solve (1.1). Then $u \in C^{p,\gamma}(\bar{\Omega})$. ∎

3. A METHOD FOR APPROXIMATING ANALYTIC OR HARMONIC FUNCTIONS

The connection between solutions of (1.1) and analytic functions shown by lemma 2.1 motivates the present section. We summarize an approximation method for analytic functions that is described in more detail in [8].

Lemma 3.1

Let Γ be a smooth curve exterior to $\partial\Omega$, and let S denote the doubly connected region between Γ and $\partial\Omega$. Let v be a solution of

$$(3.1) \qquad \begin{cases} \Delta v = 0 & \text{in } S \\ v = 1 & \text{on } \partial\Omega \\ v = 0 & \text{on } \Gamma \end{cases}$$

Then

$$(3.2) \qquad c := \int_{\partial\Omega} \frac{\partial v}{\partial n}(z)|dz| = -\int_{\Gamma} \frac{\partial v}{\partial n}(z)|dz| > 0 ,$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative, outward w.r.t. S. Let the point set $\{z_{k,n}\}_{k=1}^{n}$ be equidistributed w.r.t. $c^{-1}\frac{\partial v}{\partial n}$ on $\partial\Omega$, i.e. $z_{1,n}$ is an arbitrary point on $\partial\Omega$, and the $z_{j,n}$, $j \geq 2$, are determined by

$$(3.3) \qquad \int_{z_{1,n}}^{z_{j,n}} c^{-1}\frac{\partial v}{\partial n}(z)|dz| = \frac{j-1}{n}, \quad j = 1(1)n ,$$

where integration is done counter-clockwise along $\partial\Omega$. Let the point set $\{w_{k,n}\}_{k=1}^{n-1}$ be equidistributed w.r.t. $-c^{-1}\frac{\partial v}{\partial n}$ on Γ. Let $\psi(z)$ be a function analytic on and interior to the level curve $\Gamma_{\mu} := \{z : v(z) = \mu\}$ for some μ, $0 < \mu < 1$. Finally let $r_n(z)$ be the function in

$$(3.4) \qquad Q_n := \text{span}\{1,(z - w_{1,n})^{-1},(z - w_{2,n})^{-1},\ldots,(z - w_{n-1,n})^{-1}\}$$

uniquely determined by interpolating $\psi(z)$ at the nodes $\{z_{k,n}\}_{k=1}^{n}$. Then

$$\overline{\lim_{n\to\infty}} \sup_{z\in\partial\Omega} |\psi(z) - r_n(z)|^{1/n} \leq e^{-2\pi(1-\mu)} .$$

Proof. A proof based on results of Walsh [10] is presented in [8]. ∎

Assume that the nodes $\{z_{k,n}\}_{k=1}^{n}$ are given on $\partial\Omega$ and ennumerated so that $z_{k,n}$ precedes $z_{k+1,n}$, $k = 1(1)n - 1$, when $\partial\Omega$ is traversed counterclockwise. We wish to determine a set of poles $\{w_{k,n}\}_{k=1}^{n-1}$ equidistributed w.r.t. $-c^{-1}\frac{\partial v}{\partial n}$ on a level curve of v. Assume that $c^{-1}\frac{\partial v}{\partial n}$ is a piecewise constant function with jumps at the $z_{k,n}$. The assumption that $\{z_{k,n}\}_{k=1}^{n}$ be equidistributed w.r.t. $c^{-1}\frac{\partial v}{\partial n}$ and $c^{-1}\int_{\partial\Omega}\frac{\partial v}{\partial n}(z)|dz| = 1$ determines $\frac{\partial v}{\partial n}$ uniquely on $\partial\Omega$. This suggests that we determine level curves Γ and point sets $\{w_{k,n}\}_{k=1}^{n-1}$ by solving (3.1) as an initial value problem for v, with v and $\frac{\partial v}{\partial n}$ known on $\partial\Omega$. A simple numerical method for solving this initial value problem is described in [8] and computed examples are given in Section 5. Here we only note that we do not need to solve the initial value problem to high accuracy, and we generally do not put Γ very far from $\partial\Omega$. Therefore we can obtain a satisfactory approximate solution to this ill-posed problem. We next turn to the choice of basis for the representation of $r_n(z)$. The basis implicit in the definition (3.4) of Q_n is generally ill-conditioned. In [8] it is shown that the following basis functions are fairly well-conditioned

$$(3.5)\quad \begin{cases} \ell_0(z) := 1 \\ \ell_k(z) := \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{z - z_{j,n}}{z_{k,n} - z_{j,n}} \prod_{j=1}^{n-1} \frac{z_{k,n} - w_{j,n}}{z - w_{j,n}}, \quad k = 1(1)n - 1 . \end{cases}$$

This basis we will use when computing eigenvalues and eigenfunctions.

4. THE COLLOCATION METHOD

The eigenvalue problem (1.1), we solve as a sequence of Dirichlet problems, i.e. for given λ we solve

$$\begin{cases} \Delta u + \lambda u = 0 & \text{(4.1a)} \\ u = 0 & \text{(4.1b)} \end{cases}$$

for u and determine iteratively values of λ for which (4.1) has nontrivial solutions u. The iteration method has previously been used by Moler [5]. Equations (4.1) are solved by boundary collocation. Let $\{z_{k,n}\}_{k=1}^{n}$ and $\{w_{k,n}\}_{k=1}^{n-1}$ be point sets on $\partial\Omega$ and Γ, respectively, as described in Section 3. Let the set of collocation points $\{z^c_{k,m}\}_{k=1}^{m}$, $m = 2n - 1$, be distributed equidistantly on $\partial\Omega$ w.r.t. to the same density function as $\{z_{k,n}\}_{k=1}^{n}$. Introduce the particular solutions of (4.1a),

$$\begin{cases} V_0(z,\lambda) := \mathrm{Re}\Big(1 - 0.5\lambda(\bar{z}-\bar{z}_{1,n}) \displaystyle\int_{z_{1,n}}^{z} \frac{J_1(\sqrt{\lambda(z-\zeta)(\bar{z}-\bar{z}_1)})}{\sqrt{\lambda(z-\zeta)(\bar{z}-\bar{z}_1)}}\, d\zeta\Big) \\ V_{2j}(z,\lambda) := \mathrm{Re}\Big(\ell_j(z) - 0.5\lambda(\bar{z}-\bar{z}_{1,n}) \displaystyle\int_{z_{1,n}}^{z} \ell_j(\zeta) \frac{J_1(\sqrt{\lambda(z-\zeta)(\bar{z}-\bar{z}_1)})}{\sqrt{\lambda(z-\zeta)(\bar{z}-\bar{z}_1)}}\, d\zeta\Big) \\ \qquad\qquad j = 1(1)n-1\,, \\ V_{2j-1}(2,\lambda) := \mathrm{Im}\Big(\ell_j(z) - 0.5\lambda(\bar{z}-\bar{z}_{1,n}) \displaystyle\int_{z_{1,n}}^{z} \ell_j(\zeta) \frac{J_1(\sqrt{\lambda(z-\zeta)(\bar{z}-\bar{z}_1)})}{\sqrt{\lambda(z-\zeta)(\bar{z}-\bar{z})}}\, d\zeta\Big) \\ \qquad\qquad j = 1(1)n-1\,, \end{cases} \tag{4.2}$$

where the functions $\ell_j(z)$ are defined by (3.5). We require the boundary condition to be satisfied at the collocation points. This yields the system of equation

$$(4.3) \qquad \sum_{j-1}^{2n-1} \alpha_j V_j(z^c_{k,m},\lambda) = 0, \qquad k = 1(1)m, \quad m = 2n-1 .$$

We seek λ such that (4.3) has a nontrivial solution $\underline{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_{2n-1})^T$, or equivalently, we want the matrix

$$(4.4) \quad A := [a_{kj}], \quad a_{kj} := V_j(z^c_{k,m},\lambda), \quad j = 1(1)2n-1, \quad k = 1(1)m, \quad m = 2n-1 ,$$

to be singular. We achieve this by computing the singular value decomposition of $A = A(\lambda)$,

$$A = U\Sigma V^H ,$$

where U,V are orthonormal $(2n-1) \times (2n-1)$ matrices and $\Sigma = \mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_{2n-1})$, with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{2n-1} \geq 0$. We determine λ such that $\sigma_{2n-1} = \sigma_{2n-1}(\lambda)$ vanishes, i.e. first we tabulate $\lambda \to \sigma_{2n-1}(\lambda)$ to find a local minimum, and then by fitting quadratics we minimize $\sigma^2_{2n-1}(\lambda)$. ($\sigma^2_{2n-1}(\lambda)$ is a smooth function of λ while $\sigma_{2n-1}(\lambda)$ is not). Let λ^* denote a value of λ corresponding to a local minimum of $\sigma_{2n-1}(\lambda)$. λ^* is an approximate eigenvalue of (4.1). A method to estimate or bound the error in λ^* is discussed below. Following Moler [5], we note that λ^* satisfies

$$\min_{\lambda} \min_{\|\underline{\alpha}\|_2=1} \|A(\lambda)\underline{\alpha}\|_2 = \min_{\|\underline{\alpha}\|_2=1} \|A(\lambda^*)\underline{\alpha}\|_2$$

and that $\underline{\alpha}$ is the last column of V. Hence,

$$(4.5) \qquad u(z) = \sum_{1}^{2n-1} \alpha_j V_j(z,\lambda^*)$$

is an approximate eigenfunction. If also $\sigma_{2n-2}(\lambda^*)$ is near 0, then λ^*

is an approximate double eigenvalue. The coefficients of the other eigenfunction are in the next to last column of V.

The integrals in (4.2), we compute by integrating along $\partial\Omega$ and using a Lobatto rule between each pair of collocation points $((z^c_{k,m}), (z^c_{k+1,m}))$. We note that the functions $\ell_j(z)$ have to be computed only once at the Lobatto abscissae. The minimization scheme for $\sigma^2_{2n-1}(\lambda)$ is designed so that we in each iteration know values λ_ℓ, λ_r with $\lambda_\ell < \lambda^* < \lambda_r$. As the iterations proceed $\lambda_r - \lambda_\ell \to 0$. This enables the computation of the Bessel functions in the integrands (4.2) by linear interpolation when $0 < \lambda_r - \lambda_\ell$ is small.

Remark 4.1

In order to facilitate the detection of eigenvalues the basis functions used should be well-conditioned in the sense that $\sigma_{2n-1}(\lambda) >> 0$ for λ between local minima of $\sigma_{2n-1}(\lambda)$. Moreover, for $\lambda = \lambda^*$, a local minimum of $\sigma_{2n-1}(\lambda)$, there should be a k such that

$$\sigma_k(\lambda^*) >> \sigma_{k+1}(\lambda^*) \approx \sigma_{k+2}(\lambda^*) \approx \ldots \approx \sigma_{2n-1}(\lambda^*) \approx 0 ,$$

so that the multiplicity of the eigenvalue can be determined easily. In all computed examples the basis (4.2) satisfied these requirements. ■

Remark 4.2

The allocation of collocation points $z^c_{k,m}$ can be done quite arbitrarily. In all examples we have distributed the collocation points equidistantly w.r.t. arc length. ■

Remark 4.3

The choice of function space for the approximation of eigenfunctions is based on lemma 3.1, which discusses approximation of analytic functions by certain rational function. By taking real and imaginary parts, we obtain the harmonic functions $V_1(z,0),\ldots,V_{2n-1}(z,0)$. Approximation of harmonic functions by boundary collocation has so far been studied only when the approximating

functions are harmonic polynomials, see Curtiss [2]. These results suggest that the collocation points $\{z^c_{k,m}\}_{k=1}^m$ should be used when approximating by function (4.2) with $\lambda = 0$. For $\lambda > 0$ no results are known, and we expect for large values of λ least squares collocation to yield better approximations. In least squares collocation, see Moler [6], one chooses $m > 2n - 1$ in (4.3) and solves (4.3) in the least squares sense. Again this leads to the singular value decomposition of the matrix (4.4), but now one seeks local minima of $\lambda \to \sigma^2_{2n-1}(\lambda) \cdot \left(\iint_\Omega u^2 dxdy\right)^{-1}$, where u is defined by (4.5). Numerical experiments indicate that, indeed, the approximation errors in the eigenvalues and eigenfunctions decrease when least square collocation is used. However, the decrease in error was small, and in all computed examples a more efficient way to reduce the error was to increase the number of basis functions and let $m = 2n-1$. In the numerical examples referred to, $\partial\Omega$ was analytic and $\lambda \lesssim 10$. ∎

We conclude this section with some comments on the computation of error bounds as described in Moler and Payne [7]. Let λ^* be a real nonnegative number and define $u^*(z)$ as

$$u^*(z) := \sum_{j=1}^{n} \alpha_j v_j(z,\lambda^*) .$$

Let λ_k be the eigenvalue of (1.1) closest to λ^*, and let $u_k(z)$ be the eigenfunction belonging to λ_k. Moler and Payne [7] show how bounds for $\lambda^* - \lambda_k$ and $u^* - u_k$ can be computed. We only give their result on bounds for $\lambda^* - \lambda_k$.

Lemma 4.1

Let $\varepsilon := \max_{z\in\partial\Omega} |u^*(z)| \cdot (\text{area of } \Omega)^{1/2} \cdot \left(\iint_\Omega (u^*)^2 dxdy\right)^{-1}$. Then there is an eigenvalue of λ_k of (4.1) in the interval $\frac{\lambda^*}{1+\varepsilon} \le \lambda_k \le \frac{\lambda^*}{1-\varepsilon}$. ∎

The double integral $\iint_{\Omega} (u^*)^2 dxdy$ we compute by numerical quadrature, and we neglect round-offs during all computations. We therefore only compute error estimates. The integration rule of the next lemma was chosen because of its simplicity.

Lemma 4.2

Let $I := \{(j,k) : \{(x,y) : |jh - x| \leq \frac{h}{2}, |kh - y| \leq \frac{h}{2}\} \subset \bar{\Omega}\}$. Then for a sufficiently smooth function u which vanishes on $\partial\Omega$,

$$\left|\iint_{\Omega} u^2(x,y)dxdy - h^2 \sum_{(j,k)\in I} u^2(jh,kh)\right| \leq Ch^3 ,$$

where C is a constant.

Proof. See appendix. ∎

5. NUMERICAL EXAMPLES

In all examples the collocation points $\{z^c_{k,m}\}_{k=1}^{m}$, $m = 2n-1$, and the points $\{z_{k,n}\}_{k=1}^{n}$ in the definition of the function (3.5) are equidistributed on $\partial\Omega$ w.r.t. arc length. By solving (3.1) as an initial value problem for v and $\frac{\partial v}{\partial u}$ on $\partial\Omega$ we compute approximate level curves $\tilde{\Gamma}$ of v and point sets $\{w_{k,n}\}_{k=1}^{n-1}$ which are approximately equidistributed w.r.t. $-c^{-1}\frac{\partial v}{\partial n}$ on $\tilde{\Gamma}$. Below we show graphs with Ω, curves $\tilde{\Gamma}$ and points $\{w_{k,n}\}_{k=1}^{n-1}$. In all examples the integrals in (4.2) have been computed by integrating along $\partial\Omega$ and using a 10-point Lobatto rule between each pair of collocation points $(z^c_{j,m}, z^c_{j+1,m})$, $j = 1(1)m-1$. The double integrals in the error bound of Lemma 4.1 have been computed by the method of Lemma 4.2. We have computed $\iint_\Omega u^2 dxdy$ for different values of h, and decreased h until we were certain that the value of the integral was known with 2 significant digits. All computations were done on a VAX 11/780 in double precision, i.e. with 12 significant digits. In the first example, we determine eigenvalues for an elliptic membrane for which the lowest eigenvalue is known with high accuracy.

Ex. 5.1 $\Omega := \{(x,y) : \frac{1}{4}x^2 + y^2 < 1\}$. Figure 5.1 shows Ω, approximate level curves $\tilde{\Gamma}$ and the points $w_{j,8}$, $j = 1(1)8$, at the vertices of the outermost approximate level curve. The $w_{j,8}$ are used to define 17 basis functions (4.2).

Figure 5.2 is similar to Figure 5.1, but now 16 points $w_{j,17}$, $j = 1(1)16$ are shown on the outermost approximate level curve $\tilde{\Gamma}$. The 16 points $w_{j,17}$ define 33 basic functions (4.2).

The ε in the table is defined in lemma 4.1. The table gives the 3 smallest eigenvalues determined, the smallest of which is known, see [5], to lie in the interval [3.56672658, 3.566726614].

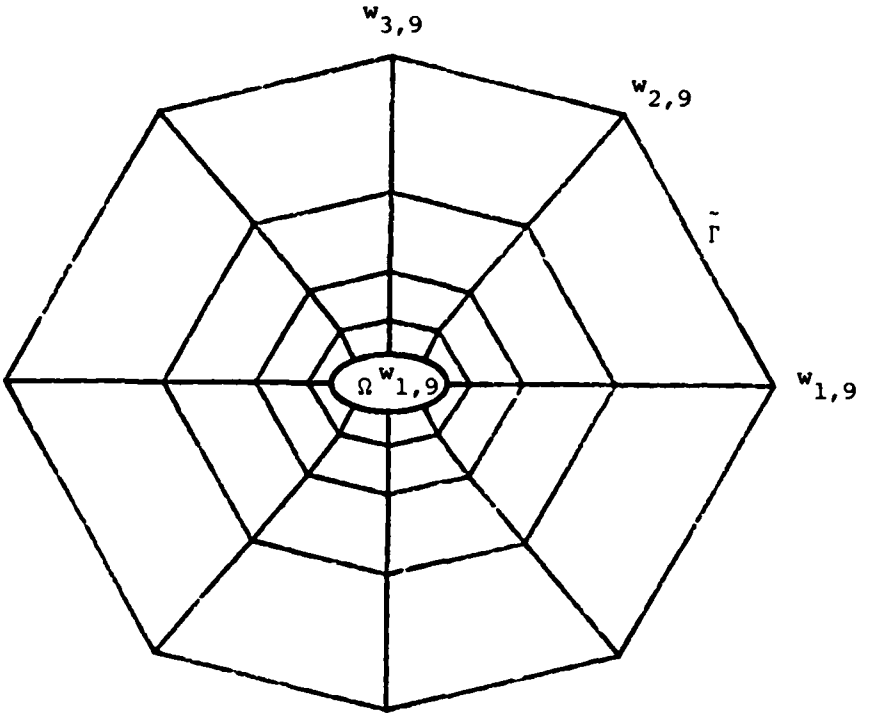

Figure 5.1: 17 basis functions

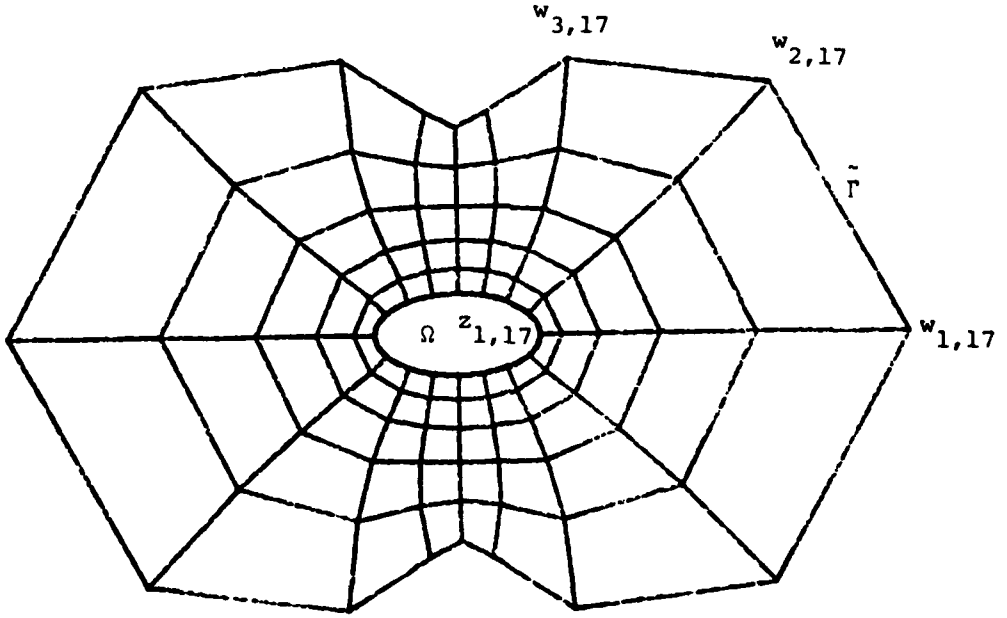

Figure 5.2: 33 basis functions

No. of Basis Facts	Computed Eigenvalue	ε	Est. Upper Bound for Eigenvalue Est. Lower Bound for Eigenvalue
17	3.56672688	$8.5 \cdot 10^{-5}$	3.5670 ------ 3.5664
33	3.566726603	$2.8 \cdot 10^{-9}$	3.56672662 ---------- 3.56672659
17	6.27535	$4.7 \cdot 10^{-3}$	6.306 ----- 6.245
33	6.275432466	$9.4 \cdot 10^{-7}$	6.275439 -------- 6.275426
17	10.034	$1.2 \cdot 10^{-1}$	11.36 ----- 8.98
33	10.028402	$2.2 \cdot 10^{-5}$	10.02863 -------- 10.02817

The error estimates generally appear to be quite conservative. ∎

Ex. 5.2 Let Ω be the interior of the curve

$$\partial\Omega := \{x(t) + iy(t),\ x(t) := 1.4\cos(t) + 2.8\cos(2t) - 2.8,$$

$$y(t) := 1.6\sin(t) + 2.45\sin(t - 0.2) + 0.98\sin(2t) -$$

$$- 0.49\sin(4t) - 0.74,\quad 0 \leq t < 2\pi\},$$

see Figure 5.3. The lowest eigenvalue has been computed with different numbers of basis functions

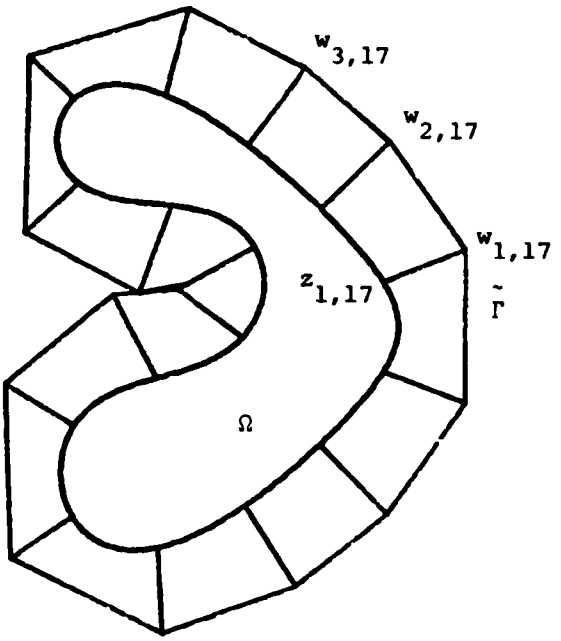

Figure 5.3: 33 basis functions

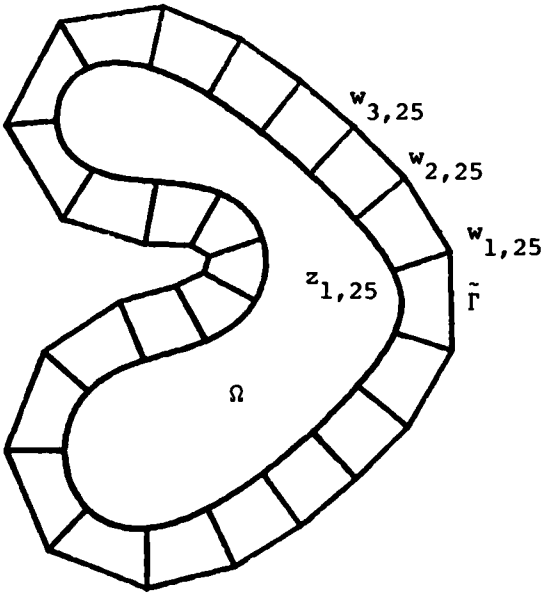

Figure 5.4: 49 basis functions

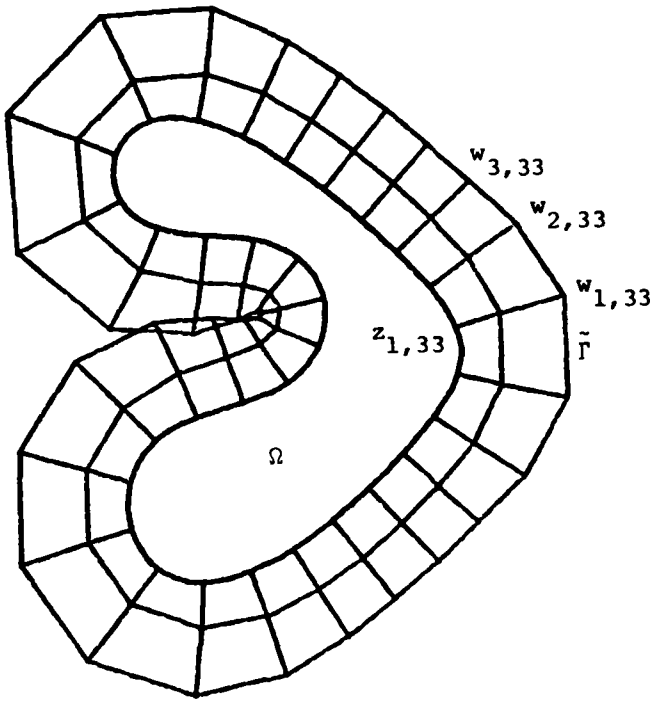

Figure 5.5: 65 basis functions

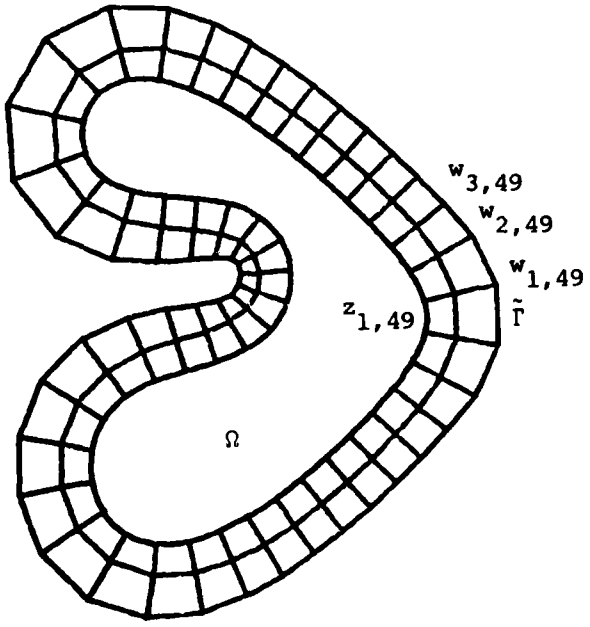

Figure 5.6: 97 basis functions

No. of Basis Facts	Computed Eigenvalue	ε	Est. Upper Bound for Eigenvalue Est. Lower Bound for Eigenvalue
33	1.2259	2.4	
49	1.22980	1.4	
65	1.2305134	$8.4 \cdot 10^{-3}$	$\frac{1.241}{1.220}$
97	1.23053147	$3.8 \cdot 10^{-3}$	$\frac{1.2353}{1.2252}$

The points $w_{j,n}$ for the different bases are shown in Figures 5.3-5.6. The allocation of poles w_j for 33 and 65 basis functions is similar, and so is the allocation for 49 and 97 basis functions. The example shows that the allocation of poles does influence the rate of convergence, but the computed eigenvalues are not very sensitive to differences in allocation. ∎

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APPENDIX

Proof of lemma 4.2

We assume for simplicity that $u \in C^4(\bar{\Omega})$. Let S be the square $\{(x,y) : |x| \leq \frac{h}{2}, |y| \leq \frac{h}{2}\}$ and let $f = u^2$. Then, Engels [4], ch. 8,

$$Tf := \iint_S f(x,y)dxdy =$$

$$= \frac{h^2}{6}\,(2f(0,0) + f(h,0) + f(-h,0) + f(0,h) + f(0,-h)) + 0(h^6) .$$

This yields

$$Tf := h^2 f(0,0) + h^4\Big(\frac{\partial^2 f}{\partial x^2}\,(0,0) + \frac{\partial^2 f}{\partial y^2}\,(0,0)\Big) + 0(h^6) .$$

Let $\hat{S} := \bigcup_{(j,k)\in I} \{x,y) : |jh - x| \leq \frac{h}{2}, |kh - g| \leq \frac{h}{2}\}$. Then

$$\iint_\Omega f(x,y)dxdy = \iint_{\hat{S}} f(x,y)dxdy + \iint_{\Omega\setminus\hat{S}} f(x,y)dxdy =$$

$$= h^2 \sum_{(j,k)\in I} f(jh,kh) + h^4 \sum_{(j,k)\in I} \Big(\frac{\partial^2 f}{\partial x^2}\,(jh,kh) + \frac{\partial^2 f}{\partial y^2}\,(jh,kh)\Big) +$$

$$+ \iint_{\Omega\setminus\hat{S}} f(x,y)dxdy + 0(h^6) .$$

We now make use of that $f = u^2$. $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2\big(\big(\frac{\partial u}{\partial x}\big)^2 + \big(\frac{\partial u}{\partial y}\big)^2 + u\Delta u\big)$ yields

$$h^2 \sum_{(j,k)\in I} \left(\frac{\partial^2 f}{\partial x^2}(jh,kh) + \frac{\partial^2 f}{\partial y^2}(jh,kh)\right) =$$

$$= \iint_{\hat{S}} \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\, dxdy + 0(h^2) =$$

$$= 2 \iint_{\hat{S}} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u\Delta u dxdy + 0(h^2) = 2 \int_{\partial\hat{S}} u \frac{\partial u}{\partial n}\, ds + 0(h^2) ,$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative from $\hat{S}$. By definition of $\hat{S}$, the distance between $\partial\hat{S}$ and $\partial\Omega$ is $< h$. Since grad u is bounded in $\bar{\Omega}$, and $u = 0$ on $\partial\Omega$, we have $\left|\int_{\partial\hat{S}} \frac{\partial u}{\partial u}\, ds\right| \leq d_1 h$, for some constant d_1.

Similarly $\iint_{\Omega\setminus\hat{S}} u^2 dxdy \leq d_2 h^2 \iint_{\Omega\setminus\hat{S}} dxdy \leq d_3 h^3$, for some constants d_2, d_3.

This completes the proof. ∎

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A boundary collocation method is presented for the computation of eigenvalues and eigenfunctions of the Laplacian on bounded simply connected plane regions with smooth boundary. Particular emphasis is placed on the selection of an appropriate function space and its basis for the approximation of the eigenfunctions. Numerical examples are presented.

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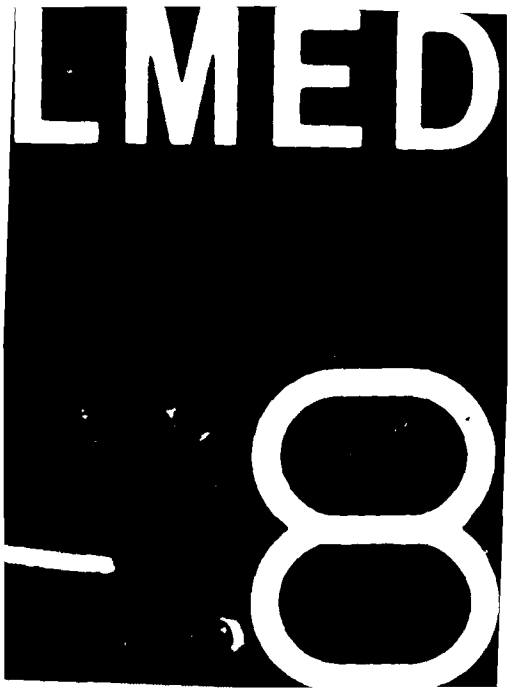
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